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Approximation and entropy numbers in Besov spaces of generalized smoothness ☆

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Dedicated to Professor Paul L. Butzer on the occasion of his 80th birthday

Abstract

We determine the exact asymptotic behaviour of entropy and approximation numbers of the limiting restriction operator $J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \to B^s_{p,q_2}(\Omega)$, defined by $J(f) = f|_\Omega$. Here Ω is a non-empty bounded domain in $\mathbb{R}^d$, ψ is an increasing slowly varying function, $0 < p < \infty$, $0 < q_1, q_2 \leqslant \infty$, $s \in \mathbb{R}$, and $B^{s,\psi}_{p,q_1}(\mathbb{R}^d)$ is the Besov space of generalized smoothness given by the function $t^s\psi(t)$. Our results improve and extend those established by Leopold [Embeddings and entropy numbers in Besov spaces of generalized smoothness, in: Function Spaces, Lecture Notes in Pure and Applied Mathematics, vol. 213, Marcel Dekker, New York, 2000, pp. 323–336].

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1. Introduction

Besov spaces $B^s_{p,q}$ appear in a natural way in approximation theory. They play an important role in rational and spline approximation as one can see in the monographs by Butzer and Berens

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[4], Bergh and Löfström [2], Peetre [24], Petrushev and Popov [25], and the references given there.

During the last 30 years, the complete solution of some natural questions has required the introduction of Besov spaces of generalized smoothness $B^{\phi}_{p,q}$. These spaces are defined similar to $B^{s}_{p,q}$ but replacing the regularity index s by a certain function ϕ with given properties. This change allows to modify the smoothness properties of the space. When $\phi(t) = t^s(1 + |\log t|)^b$, $s, b \in \mathbb{R}$, we write $B^{s,b}_{p,q}$. Note that if $b = 0$ we recover the usual spaces $B^{s}_{p,q}$.

Besov spaces of generalized smoothness have been investigated by many authors in different contexts. We refer to the papers by Merucci [23] and by Cobos and Fernandez [8] for their connection with interpolation theory. In the paper by Farkas and Leopold [12] one can find historical remarks and many other references, some of them related to the role that spaces of generalized smoothness play in probability theory and in the theory of stochastic processes. Besov spaces $B^{\phi}_{p,q}$ are also of interest in fractal analysis and the related spectral theory. At this point, we refer to the monographs by Triebel [28,30] and the references given there.

Our motivation for dealing with Besov spaces of generalized smoothness comes from the investigation of compactness of limiting embeddings. Let $\Omega \subset \mathbb{R}^d$ be a non-empty bounded domain with sufficiently smooth boundary, and let $0 < p_1, p_2, q_1, q_2 \leqslant \infty$, $-\infty < s_2 < s_1 < \infty$ with $s_1 - s_2 > d \max(1/p_1 - 1/p_2, 0)$. It is known that the embedding $id_B : B^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ is compact and its entropy numbers satisfy

$$e_k(id_B) \sim k^{-(s_1-s_2)/d} \tag{1}$$

(see [11]; terminology and notation are explained in Section 2). The behaviour of approximation numbers has been determined (see [11]) as well. In the limit case $s_1 - d/p_1 = s_2 - d/p_2$ with $s_1 \geqslant s_2$ and $0 < q_1 \leqslant q_2 \leqslant \infty$, the embedding is continuous but not compact. To overcome this obstruction, Leopold [22] modified the smoothness of the initial space, replacing $B^{s_1}_{p_1,q_1}(\Omega)$ by $B^{s_1,b}_{p_1,q_1}(\Omega)$. He showed that the embedding $id_B : B^{s_1,b}_{p_1,q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2,q_2}(\Omega)$ is compact provided that $b > \max(1/q_1 - 1/q_2, 0)$. As for the behaviour of entropy numbers in the special case $s_1 = s_2$ and $p_1 = p_2$, he proved that $e_k(id_B) \sim (\log k)^{-b}$ if $q_1 \leqslant q_2$ and $b > 0$. However, if $q_1 > q_2$ and $b > 1/q_2 - 1/q_1$, he only established the estimate

$$(\log k)^{-b} \lesssim e_k(id_B) \lesssim (\log k)^{-b+1/q_2-1/q_1}, \tag{2}$$

leaving open the exact behaviour of entropy numbers.

We shall show here that for any non-empty bounded domain $\Omega \subset \mathbb{R}^d$ the entropy numbers of the restriction operator $J : B^{s,b}_{p,q_1}(\mathbb{R}^d) \to B^{s}_{p,q_2}(\Omega)$, defined by $J(f) = f|_\Omega$, behave asymptotically like

$$e_k(J) \sim (\log k)^{-b+1/q_2-1/q_1}.$$

In particular, if Ω is sufficiently smooth so that there is an extension operator from $B^{s,b}_{p,q_1}(\Omega)$ into $B^{s,b}_{p,q_1}(\mathbb{R}^d)$, one has the same result for the embedding $id_B : B^{s,b}_{p,q_1}(\Omega) \hookrightarrow B^{s}_{p,q_2}(\Omega)$. Consequently, the exact behaviour of the entropy numbers in (2) is given by the upper bound, $e_k(id_B) \sim (\log k)^{-b+1/q_2-1/q_1}$.

In addition we shall prove that the approximation numbers of J behave in the same way as its entropy numbers. Furthermore, our techniques of proof also apply to the more general case $J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \to B^{s}_{p,q_2}(\Omega)$ with ψ being an increasing slowly varying function; we derive sharp results on approximation and entropy numbers in this situation, too.

We start by establishing a result on equivalence of approximation and entropy numbers for abstract operators between quasi-Banach spaces. This result is of independent interest. Then we apply it to embeddings between certain vector-valued sequence spaces, and finally we derive the results on function spaces by using wavelet bases.

The paper is organized as follows. In Section 2 we recall some general facts on entropy numbers, approximation numbers, and Besov spaces $B^{\phi}_{p,q}$. Equivalence results between entropy and approximation numbers of abstract operators will be given in Section 3. Section 4 contains the results on approximation and entropy numbers of embeddings between sequence spaces and between function spaces.

2. Preliminaries

Subsequently, given two sequences (b_k), (d_k) of non-negative real numbers we write $b_k \lesssim d_k$ if there is a constant $c > 0$ such that $b_k \leqslant c d_k$ for all $k \in \mathbb{N}$, while $b_k \sim d_k$ means $b_k \lesssim d_k \lesssim b_k$. For non-negative functions ϕ and ψ on $(0, \infty)$ the notation $\phi \sim \psi$ has a similar meaning; there are positive constants c_1 and c_2 such that $c_1\phi(t) \leqslant \psi(t) \leqslant c_2\phi(t)$ for all $t > 0$.

Let X and Y be quasi-Banach spaces and let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator from X into Y. For $k \in \mathbb{N}$, the kth *approximation number* $a_k(T)$ of T is defined by

$$a_k(T) = \inf \{\|T - R\| : R \in \mathcal{L}(X, Y) \text{ with rank } R < k\}.$$

Here rank R is the dimension of the range of R.

The kth (*dyadic*) *entropy number* $e_k(T)$ of T is the infimum of all $\varepsilon > 0$ such that there are $y_1, \ldots, y_q \in Y$ with $q \leqslant 2^{k-1}$ for which

$$T(B_X) \subseteq \bigcup_{j=1}^{q} (y_j + \varepsilon B_Y)$$

holds, where B_X, B_Y are the closed unit balls of X and Y, respectively (see [26,6,11]).

Clearly, the following monotonicity property holds

$$\|T\| = a_1(T) \geqslant a_2(T) \geqslant \cdots \geqslant 0 \quad \text{and} \quad \|T\| \geqslant e_1(T) \geqslant e_2(T) \geqslant \cdots \geqslant 0.$$

Moreover, entropy and approximation numbers are multiplicative, that is, for all $k, l \in \mathbb{N}$

$$a_{k+l-1}(S \circ T) \leqslant a_k(S) a_l(T), \quad e_{k+l-1}(S \circ T) \leqslant e_k(S) e_l(T).$$

Note that T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$. The asymptotic decay of the sequence $(e_k(T))$ can be considered as a measure of the "degree of compactness" of T. However, there are compact operators T such that $\lim_{k\to\infty} a_k(T) > 0$ (see [11]).

Several authors have investigated the relationships between entropy and approximation numbers (see [11, Section 1.3.3]). In the next section we shall show a new result in this direction.

Although we do not deal here with eigenvalues, it is worth to recall that there are close connections between these quantities and eigenvalues, which are the basis of many applications. Let X be a (complex) quasi-Banach space and let $T \in \mathcal{L}(X, X)$ be a compact operator. We denote by $(\lambda_k(T))$ the sequence of eigenvalues of T, counted according to their multiplicity and ordered by decreasing modulus. If T has only a finite number of eigenvalues and M is the sum of their multiplicities, then we put $\lambda_k(T) = 0$ for all $k > M$. The celebrated Carl–Triebel inequality establishes that $|\lambda_k(T)| \leqslant \sqrt{2} e_k(T)$, $k \in \mathbb{N}$ (see [5,7,11]). This inequality and estimate (1) were the starting points for Edmunds and Triebel in [11] to investigate eigenvalue distributions of degenerate elliptic differential and pseudodifferential operators. As for approximation numbers, it was shown by König (see [14]) that if X is a Banach space then $|\lambda_k(T)| = \lim_{m\to\infty} \sqrt[m]{a_k(T^m)}$.

Next we recall the definitions of the function spaces that we shall need in the sequel. We work on the d-dimensional Euclidean space $\mathbb{R}^d$ and on bounded domains Ω, that is, bounded open subsets of $\mathbb{R}^d$.

By $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ we denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. The symbol $\mathcal{F}$ stands for the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$, and $\mathcal{F}^{-1}$ for the inverse Fourier transform. Let φ_0 be a $\mathcal{C}^\infty$ function on $\mathbb{R}^d$ with

$$\varphi_0(x) = 1 \text{ if } |x| \leqslant 1 \quad \text{and} \quad \operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^d : |x| < 2\}.$$

For $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$, put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. Then one has a dyadic resolution of unity, $\sum_{k=0}^\infty \varphi_k(x) = 1$ for all $x \in \mathbb{R}^d$.

By $\mathcal{B}$ we denote the class of all continuous functions $\phi : (0,\infty) \longrightarrow (0,\infty)$ with $\phi(1) = 1$ and such that

$$\bar{\phi}(t) = \sup_{u>0} \frac{\phi(tu)}{\phi(u)} < \infty \quad \text{for every } t > 0.$$

For $\phi \in \mathcal{B}$ and $0 < p, q \leqslant \infty$, the *Besov space of generalized smoothness* $B^\phi_{p,q}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ having a finite quasi-norm

$$\|f\|_{B^\phi_{p,q}(\mathbb{R}^d)} = \left(\sum_{j=0}^\infty \left(\phi(2^j) \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L_p} \right)^q \right)^{1/q}$$

with the usual modification if $q = \infty$. Note that this definition also makes sense if ϕ is only equivalent to a function in $\mathcal{B}$.

Moreover, as an immediate consequence of Hölder's inequality, we have the following result: Given $\phi_1, \phi_2 \in \mathcal{B}$, $0 < p < \infty$ and $0 < q_1, q_2 \leqslant \infty$, and defining $1/q = \max(1/q_2 - 1/q_1, 0)$, there is a bounded embedding

$$B^{\phi_1}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^{\phi_2}_{p,q_2}(\mathbb{R}^d) \quad \text{if and only if} \quad \left(\phi_2(2^j)/\phi_1(2^j) \right) \in \ell_q. \tag{3}$$

Spaces $B^\phi_{p,q}(\mathbb{R}^d)$ were considered in [23,8,1], among other papers. For $\phi(t) = t^s$ with $s \in \mathbb{R}$, we recover the usual Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, see [4,24,27,28,30]. A case that will be of special interest for us is when $\phi(t) \sim t^s \psi(t)$, where ψ is a slowly varying function and $s \in \mathbb{R}$. Recall that a Lebesgue measurable function $\psi : (0,\infty) \longrightarrow (0,\infty)$ is said to be *slowly varying* if

$$\lim_{t\to\infty} \frac{\psi(ut)}{\psi(t)} = 1 \quad \text{for all } u > 0$$

(see [3,10]). When $\phi(t) \sim t^s \psi(t)$ we write $B^{s,\psi}_{p,q}(\mathbb{R}^d)$ instead of $B^\phi_{p,q}(\mathbb{R}^d)$. If $\psi(t) = (1 + |\log t|)^b$ with $b \in \mathbb{R}$, we simply put $B^{s,b}_{p,q}(\mathbb{R}^d)$. Spaces $B^{s,b}_{p,q}(\mathbb{R}^d)$ arise by extrapolation procedures within the scale of usual Besov spaces (see [9]).

A related, but more restrictive concept than slow variation is due to Triebel (see [28, Section 22.2]). He called a function $\Psi : (0, 1] \longrightarrow (0, \infty)$ *admissible*, if Ψ is monotone and $\Psi(2^{-j}) \sim \Psi(2^{-2j})$. In this case the function

$$\varrho_\Psi(t) = \begin{cases} \Psi(t) & \text{if } 0 < t \leqslant 1, \\ \Psi(t^{-1}) & \text{if } 1 \leqslant t < \infty \end{cases}$$

is equivalent to a slowly varying function. Spaces $B^{s,\Psi}_{p,q}(\mathbb{R}^d)$, defined as $B^{\phi}_{p,q}(\mathbb{R}^d)$ with $\phi(t) = t^s \varrho_\Psi(t)$, are important in fractal analysis (see [28, Sections 22 and 23], [30, Section 1.9.5], and the references given there).

In our later considerations wavelet representations of Besov spaces will be an essential tool, allowing us to transform our problem in function spaces to the simpler context of sequence spaces. Recently this technique has been successfully applied to deal with similar problems for weighted Besov and Triebel–Lizorkin spaces (see e.g. [13,19–21,17]).

Before we briefly describe the wavelet representation of Besov spaces $B^{\phi}_{p,q}(\mathbb{R}^d)$, we introduce sequence spaces $b^{\phi}_{p,q}$ which are naturally associated to this construction. Let $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, put $L_0 = 1$ and $L_j = 2^d - 1$ if $j \in \mathbb{N}$. Given $\phi \in \mathcal{B}$ and $0 < p, q \leqslant \infty$, the space $b^{\phi}_{p,q}$ consists of all sequences $\lambda = (\lambda_{jlm})$, indexed by $j \in \mathbb{N}_0$, $1 \leqslant l \leqslant L_j$, and $m \in \mathbb{Z}^d$, such that the quasi-norm

$$\|\lambda\|_{b^{\phi}_{p,q}} = \left(\sum_j \left(\phi(2^j) 2^{-jd/p} \right)^q \sum_l \left(\sum_m |\lambda_{jlm}|^p \right)^{q/p} \right)^{1/q}$$

is finite, with the usual modification if $p = \infty$ or $q = \infty$.

It is well known that for every $r \in \mathbb{N}$ there exist compactly supported functions $\psi_0, \psi^l \in \mathcal{C}^r(\mathbb{R}^d)$, $1 \leqslant l \leqslant 2^d - 1$, satisfying the moment conditions

$$\int_{\mathbb{R}^d} x^\alpha \psi^l(x)\, dx = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leqslant r,$$

such that the system

$$\left\{ 2^{jd/2} \psi_{jlm} : j \in \mathbb{N}_0, 1 \leqslant l \leqslant L_j, m \in \mathbb{Z}^d \right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$, where the functions ψ_{jlm} are defined by

$$\psi_{jlm} = \begin{cases} \psi_0(x - m) & \text{if } j = 0,\ l = 1,\ m \in \mathbb{Z}^d, \\ \psi^l(2^{j-1}x - m) & \text{if } j \in \mathbb{N},\ 1 \leqslant l \leqslant 2^d - 1,\ m \in \mathbb{Z}^d. \end{cases}$$

For the usual spaces $B^s_{p,q}(\mathbb{R}^d)$, the following result can be found in the paper by Triebel [29], his book [30], and the references therein. The extension to Besov spaces of generalized smoothness $B^{\phi}_{p,q}(\mathbb{R}^d)$ is due to Almeida [1].

Let $\phi \in \mathcal{B}$, $0 < p < \infty$, and $0 < q \leqslant \infty$. Then there exists a number $r(\phi, p) > 0$ such that for any system $\{\psi_{jlm}\}$ as above with $r > r(\phi, p)$ the following holds: A distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^{\phi}_{p,q}(\mathbb{R}^d)$ if and only if it can be represented as

$$f = \sum_{jlm} \lambda_{jlm} \psi_{jlm} \quad \text{with } \lambda(f) := (\lambda_{jlm}) \in b^{\phi}_{p,q}, \tag{4}$$

where the series converges unconditionally in $\mathcal{S}'(\mathbb{R}^d)$ and the coefficients are uniquely determined by

$$\lambda_{jlm} = 2^{jd}\langle f, \psi_{jlm}\rangle. \tag{5}$$

Moreover, $\|\lambda(f)\|_{b^\phi_{p,q}}$ defines a quasi-norm which is equivalent to $\|f\|_{B^\phi_{p,q}(\mathbb{R}^d)}$.

Now let $\Omega \subset \mathbb{R}^d$ be a bounded domain. As usual the space $B^\phi_{p,q}(\Omega)$ is defined by restriction of $B^\phi_{p,q}(\mathbb{R}^d)$ to Ω, and equipped with the quasi-norm

$$\|f\|_{B^\phi_{p,q}(\Omega)} = \inf\left\{\|g\|_{B^\phi_{p,q}(\mathbb{R}^d)} : g \in B^\phi_{p,q}(\mathbb{R}^d),\ g|_\Omega = f\right\}. \tag{6}$$

For any (finite or countable) index set I, we denote by $\ell_p(I)$ the space of all complex sequences $y = (y_i)_{i\in I}$ with finite quasi-norm

$$\|y\|_p = \begin{cases} \left(\sum_{i\in I} |y_i|^p\right)^{1/p} & \text{if } 0 < p < \infty,\\ \sup_{i\in I} |y_i| & \text{if } p = \infty. \end{cases}$$

If $I = \mathbb{N}$ or $I = \{1, \dots, M\}$, we write simply ℓ_p or ℓ_p^M, respectively.

Finally, given $0 < p, q \leqslant \infty$, $w_j > 0$, and $M_j \in \mathbb{N}$, let $\ell_q(w_j \ell_p^{M_j})$ denote the space of all vector-valued sequences $x = (x_j)_{j\in\mathbb{N}_0}$ with $x_j \in \ell_p^{M_j}$ and

$$\|x\|_{\ell_q(w_j\ell_p^{M_j})} = \left(\sum_{j=0}^\infty (w_j\|x_j\|_p)^q\right)^{1/q} < \infty$$

(with the usual modification if $q = \infty$). If all $w_j = 1$, we simply write $\ell_q(\ell_p^{M_j})$.

3. On equivalence of approximation and entropy numbers

In general, the asymptotic behaviour of the approximation numbers of an operator may be quite different from the asymptotic behaviour of its entropy numbers. The following examples illustrate this fact.

Example 3.1. For the diagonal operator $D : \ell_2 \longrightarrow \ell_2$ (complex spaces), $Dx = (2^{-n}x_n)$ for $x = (x_n)$, one has (see [6, p. 106])

$$a_k(D) = 2^{-k} \quad \text{and} \quad e_k(D) \sim 2^{-\sqrt{k}}.$$

Example 3.2. Consider now the diagonal operator $D : \ell_1 \longrightarrow \ell_2$ defined by $Dx = ((\log(1 + n))^{-1/2}x_n)$. By [26, Theorem 11.11.7], we have

$$a_k(D) = \sup_{n\geqslant k}\left(\frac{n+1-k}{\sum_{j=1}^n \log(1+j)}\right)^{1/2} \sim (\log k)^{-1/2}$$

and according to the results in [15,16], $e_k(D) \sim k^{-1/2}$.

However, under certain assumptions entropy and approximation numbers are equivalent. Our next result gives a sufficient condition for this equivalence.

Theorem 3.3. *Let X, Y be quasi-Banach spaces and let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Assume that (b_k) is a decreasing sequence of positive real numbers satisfying that*

$$\lim_{k\to\infty} b_k = 0 \quad \textit{and} \quad b_k \sim b_{2k}. \tag{7}$$

Then the one-sided inequalities

$$a_k(T) \lesssim b_k \lesssim e_k(T) \tag{8}$$

imply the equivalence

$$a_k(T) \sim b_k \sim e_k(T). \tag{9}$$

Proof. A result of Triebel, which extends a previous inequality due to Carl (see [11, Section 1.3.3]), shows that for any positive increasing function $f : \mathbb{N} \longrightarrow \mathbb{R}$ with $f(k) \sim f(2k)$, there is a constant $C > 0$ depending only on f and the quasi-triangle constant of Y such that for all $m \in \mathbb{N}$

$$\max_{1\leqslant k\leqslant m} f(k)e_k(T) \leqslant C \max_{1\leqslant k\leqslant m} f(k)a_k(T). \tag{10}$$

Applying this inequality with $f(k) = 1/b_k$, we obtain the desired upper estimate for entropy numbers

$$\frac{1}{b_m}\, e_m(T) \leqslant \max_{1\leqslant k\leqslant m} \frac{1}{b_k}\, e_k(T) \leqslant C \max_{1\leqslant k\leqslant m} \frac{1}{b_k} a_k(T) \leqslant C'.$$

Hence, to complete the proof it suffices to prove the lower estimate for approximation numbers. By (7) there exists a constant $C_b > 1$ such that $b_k \leqslant C_b b_{2k}$ for all $k \in \mathbb{N}$. Setting $\alpha = \log_2 C_b > 0$, this implies that

$$k^\alpha b_k \leqslant C_b m^\alpha b_m \quad \text{whenever } 1 \leqslant k \leqslant m. \tag{11}$$

On the other hand, assumption (8) yields that there are positive constants C_1, C_2 such that

$$C_1 a_k(T) \leqslant b_k \leqslant C_2 e_k(T) \quad \text{for all } k \in \mathbb{N}. \tag{12}$$

Whence, applying (10) now with $f(k) = k^{\alpha+1}$, we get that, with a constant $C > 0$ depending only on α and Y,

$$\begin{aligned} \frac{b_m}{C_2} m^{\alpha+1} &\leqslant \max_{1\leqslant k\leqslant m} k^{\alpha+1} e_k(T) \leqslant C \max_{1\leqslant k\leqslant m} k^{\alpha+1} a_k(T) \\ &= C \max\left(\max_{1\leqslant k\leqslant \delta m} k^{\alpha+1} a_k(T),\ \max_{\delta m\leqslant k\leqslant m} k^{\alpha+1} a_k(T) \right), \end{aligned}$$

where $0 < \delta < 1$ will be chosen later appropriately.

Due to (11) and (12) we have for any $k \leqslant \delta m$,

$$k^{\alpha+1} a_k(T) \leqslant \delta m k^\alpha \frac{b_k}{C_1} \leqslant \frac{\delta C_b}{C_1} m m^\alpha b_m.$$

Hence, taking $\delta < C_1/(C_2 C C_b)$, we see that

$$C \max_{1\leqslant k\leqslant \delta m} k^{\alpha+1} a_k(T) < \frac{b_m}{C_2} m^{\alpha+1}.$$

This yields that

$$\frac{b_m}{C_2} m^{\alpha+1} \leqslant C \max_{\delta m \leqslant k \leqslant m} k^{\alpha+1} a_k(T) \leqslant C m^{\alpha+1} a_{[\delta m]}(T),$$

where $[\,\cdot\,]$ is the greatest integer function. Consequently,

$$a_{[\delta m]}(T) \geqslant \frac{b_m}{C_2 C} \sim b_{[\delta m]},$$

which gives $a_m(T) \gtrsim b_m$. This completes the proof. □

Note that the assumption $b_k \sim b_{2k}$ is essential in Theorem 3.3. For instance, the operator in Example 3.1 satisfies (8) with $b_k = 2^{-k}$, while (9) fails.

In the next section we shall also need, in addition to Theorem 3.3, the following recent result of the second named of the present authors [18].

Lemma 3.4. *Let $0 < q < p \leqslant \infty$ and $1/r = 1/q - 1/p$. Assume that $(\sigma_m) \in \ell_r$ is a decreasing sequence of positive real numbers and let $D_\sigma : \ell_p \longrightarrow \ell_q$ be the diagonal operator defined by $D_\sigma(x_m) = (\sigma_m x_m)$. Put $w_m = (\sum_{k=m}^{\infty} \sigma_k^r)^{1/r}$. If $w_m \sim w_{2m}$ then $e_m(D_\sigma) \sim w_m$.*

In the Banach case $1 \leqslant q < p \leqslant \infty$, Pietsch showed that $a_m(D_\sigma) = w_m$, even without the doubling condition $w_m \sim w_{2m}$ (see [26, Theorem 11.11.4]). However, Lemma 3.4 does not remain true without that assumption (see [18]).

4. Compact embeddings

Our aim is to prove asymptotically sharp estimates for approximation and entropy numbers of the restriction operator

$$J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \longrightarrow B^s_{p,q_2}(\Omega) \quad \text{defined by } Jf = f|_\Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, ψ an increasing slowly varying function, $0 < p < \infty$, $0 < q_1, q_2 \leqslant \infty$, and $s \in \mathbb{R}$.

For spaces on $\mathbb{R}^d$ we have, as a special case of the general result (3), that there is a bounded embedding $B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^d)$ if and only if

$$\left(\psi(2^j)^{-1}\right) \in \ell_q \quad \text{where } 1/q = \max(1/q_2 - 1/q_1, 0). \tag{13}$$

Observe that this condition is also equivalent to the existence of a bounded embedding id : $\ell_{q_1}(\psi(2^j)\ell_p^{M_j}) \longrightarrow \ell_{q_2}(\ell_p^{M_j})$, with arbitrary $M_j \in \mathbb{N}$.

Using the wavelet description of Besov spaces, we establish now two factorization diagrams for the operator J, which will reduce the study of J to the investigation of embeddings in certain sequence spaces. To this end we introduce the index sets

$$I_j = \{(l, m) : 1 \leqslant l \leqslant L_j,\ \text{supp}\,(\psi_{jlm}) \cap \Omega \neq \emptyset\}$$

and

$$\widetilde{I}_j = \{(l, m) : 1 \leqslant l \leqslant L_j,\ \text{supp}\,(\psi_{jlm}) \subseteq \Omega\}.$$

Since the wavelets ψ_{jlm} are compactly supported and Ω is bounded with non-empty interior, one easily verifies that

$$\operatorname{card} I_j \sim 2^{jd} \sim \operatorname{card} \widetilde{I}_j. \tag{14}$$

Now let us consider the operators

$$S : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \longrightarrow \ell_{q_1}(\psi(2^j)\ell_p(I_j)) \quad \text{and} \quad T : \ell_{q_2}(\ell_p(I_j)) \longrightarrow B^s_{p,q_2}(\Omega)$$

defined by

$$Sf = \left(2^{js+jd(1-1/p)}\langle f, \psi_{jlm}\rangle\right)_{j\in\mathbb{N}_0,(l,m)\in I_j}$$

and

$$T(\mu_{jlm}) = \sum_{j=0}^{\infty} \sum_{(l,m)\in I_j} 2^{-js-jd(1-1/p)}\mu_{jlm}\psi_{jlm}|_{\Omega}.$$

It follows from (4), (5) and the definition of the quasi-norm in $B^s_{p,q_2}(\Omega)$ that the operators S and T are bounded. If condition (13) holds, then the identity $id_b : \ell_{q_1}(\psi(2^j)\ell_p(I_j)) \longrightarrow \ell_{q_2}(\ell_p(I_j))$ is bounded and we have the following commutative diagram

$$\begin{tikzcd} B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \arrow[r, "J"] \arrow[d, "S"'] & B^s_{p,q_2}(\Omega) \\ \ell_{q_1}(\psi(2^j)\ell_p(I_j)) \arrow[r, "id_b"'] & \ell_{q_2}(\ell_p(I_j)) \arrow[u, "T"'] \end{tikzcd}.$$

Consequently, J is bounded as well, and by the multiplicativity of entropy and approximation numbers we have

$$e_k(J) \lesssim e_k(id_b) \quad \text{and} \quad a_k(J) \lesssim a_k(id_b).$$

Conversely, let the operators

$$\widetilde{T} : \ell_{q_1}(\psi(2^j)\ell_p(\widetilde{I}_j)) \longrightarrow B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \quad \text{and} \quad \widetilde{S} : B^s_{p,q_2}(\Omega) \longrightarrow \ell_{q_2}(\ell_p(\widetilde{I}_j))$$

be defined by

$$\widetilde{T}(\eta_{jlm}) = \sum_{j=0}^{\infty} \sum_{(l,m)\in\widetilde{I}_j} 2^{-js-jd(1-1/p)}\eta_{jlm}\psi_{jlm}$$

and

$$\widetilde{S}f = \left(2^{js+jd(1-1/p)}\langle f, \psi_{jlm}\rangle\right)_{j\in\mathbb{N}_0,(l,m)\in\widetilde{I}_j}.$$

Note that $\widetilde{S}$ is well defined because supp $(\psi_{jlm}) \subseteq \Omega$ for any $(l, m) \in \widetilde{I}_j$. Similar arguments as above show that $\widetilde{T}$ and $\widetilde{S}$ are bounded. Hence, if J is bounded, we conclude from the commutative diagram

$$\begin{tikzcd} \ell_{q_1}(\psi(2^j)\ell_p(\widetilde{I}_j)) \arrow[r, "\widetilde{id}_b"] \arrow[d, "\widetilde{T}"'] & \ell_{q_2}(\ell_p(\widetilde{I}_j)) \\ B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \arrow[r, "J"'] & B^s_{p,q_2}(\Omega) \arrow[u, "\widetilde{S}"'] \end{tikzcd}$$

that the identity $\widetilde{id}_b$ is bounded, i.e. condition (13) holds. Moreover we have

$$e_k(\widetilde{id}_b) \lesssim e_k(J) \quad \text{and} \quad a_k(\widetilde{id}_b) \lesssim a_k(J).$$

Summarizing our considerations, we have shown that the restriction operator J is bounded if and only if (13) holds, and in this case one has

$$e_k(\widetilde{id}_b) \lesssim e_k(J) \lesssim e_k(id_b) \quad \text{and} \quad a_k(\widetilde{id}_b) \lesssim a_k(J) \lesssim a_k(id_b). \tag{15}$$

Before we can determine the exact asymptotic behaviour of the entropy and approximation numbers of J, we need some results on diagonal operators between vector-valued sequence spaces.

Lemma 4.1. *Let $M_j \sim 2^{jd}$, $0 < p_2 \leqslant p_1 \leqslant \infty$, $0 < q_2 < q_1 \leqslant \infty$, and set $1/p = 1/p_2 - 1/p_1$, $1/q = 1/q_2 - 1/q_1$. Let $(\sigma_j) \in \ell_q$ be a decreasing sequence such that $\sigma_j \sim \sigma_{j+1}$. Then the diagonal operator*

$$D_\sigma : \ell_{q_1}(l_{p_1}^{M_j}) \longrightarrow \ell_{q_2}(l_{p_2}^{M_j}) \quad \textit{defined by } D_\sigma(x_j) = (\sigma_j M_j^{-1/p} x_j)$$

satisfies

$$a_k(D_\sigma) \lesssim b_k \quad \textit{and} \quad e_k(D_\sigma) \lesssim b_k \quad \textit{where } b_k = \left(\sum_{2^{jd} \geqslant k} \sigma_j^q \right)^{1/q}.$$

Proof. For $N \in \mathbb{N}$, we set $k_N = \sum_{j=0}^{N-1} M_j + 1$. Let $P_N : \ell_{q_1}(l_{p_1}^{M_j}) \longrightarrow \ell_{q_1}(l_{p_1}^{M_j})$ be the projection onto the first N coordinates,

$$P_N x = (x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots) \quad \text{for } x = (x_j)_{j \in \mathbb{N}_0}.$$

Since rank $P_N < k_N$, we have

$$a_{k_N}(D_\sigma) \leqslant \| D_\sigma - D_\sigma P_N \|.$$

The norm can be estimated using Hölder's inequality. We get that

$$a_{k_N}(D_\sigma) \leqslant \left(\sum_{j=N}^{\infty} \sigma_j^q \right)^{1/q} = b_{2^{Nd}}. \tag{16}$$

From $\sigma_j \sim \sigma_{j+1}$ we infer that $b_k \sim b_{2k}$, and $M_j \sim 2^{jd}$ implies $k_N \sim 2^{Nd}$. Hence (16) and the monotonicity of approximation numbers yield that $a_k(D_\sigma) \lesssim b_k$. Finally, applying (10) with $f(k) = 1/b_k$, we conclude that $e_k(D_\sigma) \lesssim b_k$. $\square$

Lemma 4.2. *Let $M_j \sim 2^{jd}$, $0 < p < \infty$, and $0 < q_1 \leqslant q_2 \leqslant \infty$. Let (σ_j) be a decreasing sequence with $\sigma_j \sim \sigma_{j+1}$. Then for the diagonal operator $D_\sigma : \ell_{q_1}(l_p^{M_j}) \longrightarrow \ell_{q_2}(l_p^{M_j})$ defined by $D_\sigma(x_j) = (\sigma_j x_j)$ one has*

$$a_k(D_\sigma) \sim e_k(D_\sigma) \sim b_k \quad \textit{where } b_k = \sup\left\{\sigma_j : 2^{jd} \geqslant k\right\}.$$

Proof. Again we have $b_k \sim b_{2k}$, and according to Theorem 3.3 it suffices to prove the inequalities $a_k(D_\sigma) \lesssim b_k$ and $e_k(D_\sigma) \gtrsim b_k$.

For the approximation numbers we use the same arguments as in the preceding proof, observing that now $a_k(D_\sigma) \leqslant \|D_\sigma - D_\sigma P_N\| = \sigma_N = b_{2^{Nd}}$, since (σ_j) is decreasing. As above we conclude that $a_k(D_\sigma) \lesssim b_k$.

It remains to show that $e_k(D_\sigma) \gtrsim b_k$. Let $R : \ell_p^{M_r} \longrightarrow \ell_{q_1}(\ell_p^{M_j})$ be the map that associates to x the sequence having all coordinates equal to 0 but the rth coordinate which is $\sigma_r^{-1} x$. Let $Q : \ell_{q_2}(\ell_p^{M_j}) \longrightarrow \ell_p^{M_j}$ be the projection onto the rth coordinate. Using the commutative diagram

$$\begin{tikzcd}
\ell_p^{M_r} \arrow[r, "id"] \arrow[d, "R"'] & \ell_p^{M_r} \\
\ell_{q_1}(\ell_p^{M_j}) \arrow[r, "D_\sigma"'] & \ell_{q_2}(\ell_p^{M_j}) \arrow[u, "Q"']
\end{tikzcd}$$

we get by multiplicativity of entropy numbers

$$e_k(id) \leqslant \|R\| e_k(D_\sigma) \|Q\| = \sigma_r^{-1} e_k(D_\sigma).$$

Take $k = M_r \sim 2^{rd}$. Since

$$e_k(id) = e_k(id : \ell_p^{M_r} \longrightarrow \ell_p^{M_r}) \sim 1$$

(see [26,11]), it follows that $e_{2^{rd}}(D_\sigma) \geqslant c\sigma_r = c b_{2^{rd}}$ for some constant $c > 0$ and all $r \in \mathbb{N}$. Using $b_k \sim b_{2k}$, we conclude that $e_k(D_\sigma) \gtrsim b_k$. $\square$

Lemma 4.3. *Let $M_j \sim 2^{jd}$, $0 < p \leqslant \infty$, $0 < q_2 < q_1 \leqslant \infty$ and $1/q = 1/q_2 - 1/q_1$. Let $(\sigma_j) \in \ell_q$ be a decreasing sequence with $\sigma_j \sim \sigma_{j+1}$. Then for the diagonal operator*

$$D_\sigma : \ell_{q_1}(l_p^{M_j}) \longrightarrow \ell_{q_2}(l_p^{M_j}) \quad \textit{defined by } D_\sigma(x_j) = (\sigma_j x_j)$$

one has

$$a_k(D_\sigma) \sim e_k(D_\sigma) \sim b_k = \left(\sum_{2^{jd} \geqslant k} \sigma_j^q\right)^{1/q}.$$

Proof. Lemma 4.1 gives $a_k(D_\sigma) \lesssim b_k$; and $\sigma_j \sim \sigma_{j+1}$ implies $b_k \sim b_{2k}$. By Theorem 3.3 it suffices therefore to show that $e_k(D_\sigma) \gtrsim b_k$. To this end choose $0 < w < \min(p, q_2)$ and put $1/u = 1/w - 1/q_2$. Let

$$D_1 : \ell_\infty(\ell_\infty^{M_j}) \longrightarrow \ell_{q_1}(\ell_p^{M_j}), \quad D_2 : \ell_{q_2}(\ell_p^{M_j}) \longrightarrow \ell_w(\ell_w^{M_j})$$

be the operators defined by

$$D_1(x_j) = \left(\sigma_j^{q/q_1} M_j^{-1/p} x_j\right), \quad D_2(x_j) = \left(\sigma_j^{q/u} M_j^{1/p-1/w} x_j\right).$$

For the entropy numbers of these operators Lemma 4.1 yields the estimates

$$e_k(D_1) \lesssim \left(\sum_{2^{jd} \geqslant k} \sigma_j^q\right)^{1/q_1} = b_k^{q/q_1} \quad \text{and} \quad e_k(D_2) \lesssim \left(\sum_{2^{jd} \geqslant k} \sigma_j^q\right)^{1/u} = b_k^{q/u}.$$

Let $D : \ell_\infty(\ell_\infty^{M_j}) \longrightarrow \ell_w(\ell_w^{M_j})$ be the composition $D = D_2 \circ D_\sigma \circ D_1$. Then

$$D(x_j) = \left(\sigma_j^{q(1/q_1+1/q+1/u)} M_j^{-1/w} x_j\right) = \left(\sigma_j^{q/w} M_j^{-1/w} x_j\right),$$

and the following diagram commutes

$$\begin{tikzcd}
\ell_\infty(\ell_\infty^{M_j}) \arrow[r, "D"] \arrow[d, "D_1"'] & \ell_w(\ell_w^{M_j}) \\
\ell_{q_1}(\ell_p^{M_j}) \arrow[r, "D_\sigma"'] & \ell_{q_2}(\ell_p^{M_j}) \arrow[u, "D_2"']
\end{tikzcd}.$$

Viewing $\ell_\infty(\ell_\infty^{M_j})$ and $\ell_w(\ell_w^{M_j})$ as ℓ_∞ and ℓ_w, respectively, the operator D can be realized as a diagonal operator from ℓ_∞ into ℓ_w with diagonal entries

$$\delta_r = \sigma_k^{q/w} M_k^{-1/w} \quad \text{for} \sum_{j=0}^{k-1} M_j < r \leqslant \sum_{j=0}^{k} M_j,$$

that means, each entry $\sigma_j^{q/w} M_j^{-1/w}$ appears M_j times in the sequence (δ_r). Our assumption $\sigma_j \sim \sigma_{j+1}$ implies $(\sum_{r=m}^\infty \delta_r^w)^{1/w} \sim (\sum_{r=2m}^\infty \delta_r^w)^{1/w}$. Therefore we can apply Lemma 3.4 to the operator D, and we get for all $N \in \mathbb{N}$ and $k_N = \sum_{j=0}^{N-1} M_j + 1$ the estimate

$$e_{k_N}(D)^w \gtrsim \sum_{r \geqslant k_N} \delta_r^w = \sum_{j \geqslant N} \sigma_j^q = b_{2^{Nd}}^q.$$

Observing that $k_N \sim 2^{Nd}$, this gives $e_k(D) \gtrsim b_k^{q/w}$.

By the multiplicativity of entropy numbers and using $b_k \sim b_{3k}$ we obtain

$$b_k^{q/w} \sim b_{3k}^{q/w} \lesssim e_{3k}(D) \leqslant e_k(D_1)\, e_k(D_\sigma)\, e_k(D_2) \lesssim b_k^{q/q_1} e_k(D_\sigma)\, b_k^{q/u}.$$

This shows $e_k(D_\sigma) \gtrsim b_k^{q(1/w-1/q_1-1/u)} = b_k$, and the proof is finished. $\square$

Now we determine the exact asymptotic behaviour of entropy and approximation numbers of the identity $id : \ell_{q_1}(\psi(2^j)\ell_p^{M_j}) \longrightarrow \ell_{q_2}(\ell_p^{M_j})$, with $M_j \sim 2^{jd}$. In view of (14), it is clear that the identities id_b and $\widetilde{id}_b$ used in the first two factorization diagrams of this section are of this form.

Theorem 4.4. *Let* $0 < p < \infty$, $0 < q_1, q_2 \leqslant \infty$, $1/q = \max(1/q_2 - 1/q_1, 0)$, *let* ψ *be an increasing slowly varying function such that* $\left(\psi(2^j)^{-q}\right) \in \ell_q$, *and let* $M_j \sim 2^{jd}$. *Then for the identity* $id : \ell_{q_1}(\psi(2^j)\ell_p^{M_j}) \longrightarrow \ell_{q_2}(\ell_p^{M_j})$ *one has*

$$a_k(id) \sim e_k(id) \sim \begin{cases} \psi(k^{1/d})^{-1} & \textit{if } q_1 \leqslant q_2, \\ \left(\int_{k^{1/d}}^{\infty} \psi(t)^{-q}\, dt/t\right)^{1/q} & \textit{if } q_1 > q_2. \end{cases}$$

Proof. Set $\sigma_j = \psi(2^j)^{-1}$ for $j \in \mathbb{N}_0$. It is easy to check that $a_k(id) = a_k(D_\sigma)$ and $e_k(id) = e_k(D_\sigma)$, where $D_\sigma : \ell_{q_1}(\ell_p^{M_j}) \longrightarrow \ell_{q_2}(\ell_p^{M_j})$ is the diagonal operator $D_\sigma(x_j) = (\sigma_j x_j)$.

If $q_1 \leqslant q_2$ we can apply Lemma 4.2. Indeed, since ψ is slowly varying, we have in particular $\psi(2t)^{-1} \sim \psi(t)^{-1}$, whence $\sigma_j \sim \sigma_{j+1}$. Moreover, using that ψ is increasing, we get

$$b_k = \sup\left\{\sigma_j : 2^{jd} \geqslant k\right\} = \sup\left\{\psi(2^j)^{-1} : 2^{jd} \geqslant k\right\} = \psi(k^{1/d})^{-1}.$$

Therefore, Lemma 4.2 implies $a_k(D_\sigma) \sim e_k(D_\sigma) \sim \psi(k^{1/d})^{-1}$.

In the case $q_1 > q_2$ we use Lemma 4.3. Since ψ is increasing and satisfies $\psi(2t) \sim \psi(t)$, we obtain that

$$b_k^q = \sum_{2^{jd} \geqslant k} \sigma_j^q = \sum_{2^{jd} \geqslant k} \psi(2^j)^{-q} \sim \sum_{2^{jd} \geqslant k} \int_{2^j}^{2^{j+1}} \psi(t)^{-q} dt/t \sim \int_{k^{1/d}}^{\infty} \psi(t)^{-q}\, dt/t.$$

Consequently, Lemma 4.3 yields

$$a_k(D_\sigma) \sim e_k(D_\sigma) \sim \left(\int_{k^{1/d}}^{\infty} \psi(t)^{-q} dt/t\right)^{1/q}.$$

The proof is finished. □

Applying Theorem 4.4 to the identities id_b and $\widetilde{id}_b$ we see that the approximation and entropy numbers of these operators are equivalent, and combining this with the inequalities (15) we immediately obtain our main result.

Theorem 4.5. *Let* Ω *be a bounded domain in* $\mathbb{R}^d$, *let* $0 < p < \infty$, $0 < q_1, q_2 \leqslant \infty$, $1/q = \max(1/q_2 - 1/q_1, 0)$, $s \in \mathbb{R}$ *and let* ψ *be an increasing slowly varying function with* $(\psi(2^j)^{-1}) \in \ell_q$. *Let the operator* $J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \longrightarrow B^s_{p,q_2}(\Omega)$ *be given by* $J(f) = f|_\Omega$. *Then*

$$a_k(J) \sim e_k(J) \sim \begin{cases} \psi(k^{1/d})^{-1} & \textit{if } q_1 \leqslant q_2, \\ \left(\int_{k^{1/d}}^{\infty} \psi(t)^{-q}\, dt/t\right)^{1/q} & \textit{if } q_1 > q_2. \end{cases}$$

In particular, the operator J *is compact if and only if*

$$\left(\psi(2^j)^{-1}\right) \in \begin{cases} c_0 & \textit{if } q_1 \leqslant q_2, \\ \ell_q & \textit{if } q_1 > q_2. \end{cases}$$

Clearly, if Ω is sufficiently smooth so that there is an extension operator from $B^{s,\psi}_{p,q_1}(\Omega)$ into $B^{s,\psi}_{p,q_1}(\mathbb{R}^d)$, then an analogous result to Theorem 4.5 holds for the embedding $id_B : B^{s,\psi}_{p,q_1}(\Omega) \hookrightarrow B^{s}_{p,q_2}(\Omega)$.

Specializing Theorem 4.5 for the case $\psi(t) = (1 + |\log t|)^b$ we can complement the results established by Leopold in [22, Theorem 3.1], by giving the exact order of decay for entropy numbers when $q_1 > q_2$ and the exact behaviour of approximation numbers for any $0 < q_1, q_2 \leqslant \infty$.

Corollary 4.6. *Let Ω, p, q_1, q_2, q, s be as in Theorem 4.5, and let $b > 1/q$. Then the operator $J : B^{s,b}_{p,q_1}(\mathbb{R}^d) \longrightarrow B^{s}_{p,q_2}(\Omega)$ is compact and*

$$a_k(J) \sim e_k(J) \sim \begin{cases} (\log k)^{-b} & \text{if } q_1 \leqslant q_2, \\ (\log k)^{-b+1/q} & \text{if } q_1 > q_2. \end{cases}$$

We can also cover the case $q_1 > q_2$, $b = 1/q$ if we modify the function ψ by inserting a double logarithmic factor.

Corollary 4.7. *Let Ω, p, q_1, q_2, q, s be as in Theorem 4.5, where $q_2 < q_1$. Let*

$$\psi(t) = (1 + |\log t|)^{1/q}(1 + |\log(1 + |\log t|)|)^b \qquad \text{with } b > 1/q.$$

Then the operator $J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \longrightarrow B^{s}_{p,q_2}(\Omega)$ is compact and

$$a_k(J) \sim e_k(J) \sim (\log\log k)^{-b+1/q}.$$

Clearly Theorem 4.5 also applies to slowly varying functions ψ which are not admissible (for the definition see the Preliminaries). We give a typical example of such functions.

Corollary 4.8. *Let Ω and p, q_1, q_2, q be as in Theorem 4.5, and let*

$$\psi(t) = \exp\left(c(\log(1+t))^{\alpha}\right) \quad \text{where } c > 0 \text{ and } 0 < \alpha < 1.$$

Then the operator $J : B^{s,\psi}_{p,q_1}(\mathbb{R}^d) \longrightarrow B^{s}_{p,q_2}(\Omega)$ is compact and

$$a_k(J) \sim e_k(J) \sim \psi(k^{1/d})^{-1}(\log k)^{(1-\alpha)/q}.$$

Of course, the list of examples could be extended. For other possible candidates we refer to [20,17], where such functions appeared as weights in weighted function spaces.

We conclude the paper by the following remark: Because we were interested in *limiting* embeddings, we worked in this paper with slowly varying functions ψ. But (apart from monotonicity) we used in the proofs only the doubling condition $\psi(t) \sim \psi(2t)$, a property which also holds for functions in the much wider class $\mathcal{B}$. That means, implicitly we have shown an analogous version of Theorem 4.5 for more general restriction operators $J : B^{\phi_1}_{p,q_1}(\mathbb{R}^d) \to B^{\phi_2}_{p,q_2}(\Omega)$ with arbitrary functions $\phi_1, \phi_2 \in \mathcal{B}$ such that $\phi(t) := \phi_2(t)/\phi_1(t)$ is decreasing and $\left(\phi(2^j)\right) \in \ell_q$. For instance, in the *non-limiting* case $\phi(t) = t^{-s}\psi(t)$ with $s > 0$ and ψ slowly varying, we obtain $a_k(J) \sim e_k(J) \sim k^{-s/d}\psi(k^{1/d})$. We leave the details to the reader.

References

[1] A. Almeida, Wavelet bases in generalized Besov spaces, J. Math. Anal. Appl. 304 (2005) 198–211.
[2] J. Bergh, J. Löfström, Interpolation Spaces, An Introduction, Springer, Berlin, 1976.
[3] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular variation, in: Encyclopedia in Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987.
[4] P.L. Butzer, H. Berens, Semi-Groups of Operators and Approximation, Springer, New York, 1967.
[5] B. Carl, Entropy numbers, s-numbers and eigenvalue problems, J. Funct. Anal. 41 (1981) 290–306.
[6] B. Carl, I. Stephani, Entropy, Compactness and the Approximation of Operators, Cambridge University Press, Cambridge, 1990.
[7] B. Carl, H. Triebel, Inequalities between eigenvalues, entropy numbers and related quantities of compact operators in Banach spaces, Math. Ann. 251 (1980) 129–133.
[8] F. Cobos, D.L. Fernandez, Hardy–Sobolev spaces and Besov spaces with a function parameter, in: Function Spaces and Applications, Lecture Notes in Mathematics, vol. 1302, Springer, Berlin, 1988, pp. 158–170.
[9] F. Cobos, L.M. Fernández-Cabrera, H. Triebel, Abstract and concrete logarithmic interpolation spaces, J. London Math. Soc. 70 (2004) 231–243.
[10] D.E. Edmunds, W.E. Evans, Hardy Operators, Function Spaces and Embeddings, Springer, Berlin, 2004.
[11] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers, Differential Operators, Cambridge University Press, Cambridge, 1996.
[12] W. Farkas, H.-G. Leopold, Characterizations of function spaces of generalized smoothness, Ann. Mat. Pura Appl. 185 (2006) 1–62.
[13] D.D. Haroske, H. Triebel, Wavelet bases and entropy numbers in weighted function spaces, Math. Nachr. 278 (2005) 108–132.
[14] H. König, Eigenvalue Distributions of Compact Operators, Birkhäuser, Basel, 1986.
[15] T. Kühn, Entropy numbers of diagonal operators of logarithmic type, Georgian Math. J. 8 (2001) 307–318.
[16] T. Kühn, Entropy numbers of general diagonal operators, Rev. Mat. Complut. 18 (2005) 479–491.
[17] T. Kühn, Entropy numbers in weighted function spaces. The case of intermediate weights, Proc. Steklov Inst. Math. 255 (2006) 159–168.
[18] T. Kühn, Entropy numbers in sequence spaces with an application to weighted function spaces, J. Approx. Theory, in press, doi: 10.1016/j.jat.2008.01.002.
[19] T. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces, Constr. Approx. 23 (2006) 61–77.
[20] T. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces II, Proc. Edinburgh. Math. Soc. 49 (2) (2006) 331–359.
[21] T. Kühn, H.-G. Leopold, W. Sickel, L. Skrzypczak, Entropy numbers of embeddings of weighted Besov spaces III. Weights of logarithmic type, Math. Z. 255 (2007) 1–15.
[22] H.-G. Leopold, Embeddings and entropy numbers in Besov spaces of generalized smoothness, in: Function Spaces, Lecture Notes in Pure and Applied Mathematics, vol. 213, Marcel Dekker, New York, 2000, pp. 323–336.
[23] C. Merucci, Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces, in: Interpolation Spaces and allied Topics in Analysis, Lecture Notes in Mathematics, vol. 1070, Springer, Berlin, 1984, pp. 183–201.
[24] J. Peetre, New Thoughts on Besov Spaces, in: Duke University Mathematics Series, Durham, 1976.
[25] P.P. Petrushev, V.A. Popov, Rational approximation of real functions, in: Encyclopedia in Mathematics and its Applications, vol. 28, Cambridge University Press, Cambridge, 1987.
[26] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[27] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.
[28] H. Triebel, The Structure of Functions, Birkhäuser, Basel, 2001.
[29] H. Triebel, A note on wavelet bases in function spaces, in: Orlicz Centenary Volume, Banach Center Publications, vol. 64, Polish Academy of Sciences, Warszawa, 2004, pp. 193–206.
[30] H. Triebel, Theory of Function Spaces III, Birkhäuser, Basel, 2006.